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Iryna Klyus[1],
Pavlo Vasylyshyn[2]
Valentyna Konovaliuk[3]

MULTIPOINT PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

[1,3]National Aviation University
1, Kosmonavta Komarova avenue, 03680, Kyiv, Ukraine
[2] Precarpathian National University named after Vasyl Stefanyk,
[2]57, Shevchenko street, Ivano-Frankivsk, 76025, Ukraine
E-mails: [1]isklyus@gmail.com; [2]pbvasylyshyn@ukr.net; [3]valentinakonovaliuk@gmail.com

Abstract. *Correctness of the problem with multipoint conditions in time variable and frequency of the spatial coordinates for the partial differential, equations, not solved with respect to the highest derivative, with pseudodifferential operators is investigated. The conditions of existence and uniqueness of the problem solution, metric theorems on lower bounds of small denominators arising in the construction of the solution of the problem are proved.*

Keywords: differential equations; multipoint conditions; pseudodifferential operators; small denominators

1. Introduction

Problems with multipoint conditions for partial differential equations are, in general, conditionally correct [1-4], and their solvability in many cases is related to the problem of small denominators.

This paper examines the problem with multipoint in time variable conditions for partial differential equations, what are not solved with respect to the highest derivative in time, with pseudodifferential operators.

2. Statement of the problem

In the region $\overline{D}=\{(t,x):t\in(0,T),x\in\Omega_p\}$, Ω_p is p-dimensional torus $(R/2\pi Z)^p$, let us consider the problem:

$$A\left(\frac{\partial}{\partial t},D\right)u\equiv A_n(D)\frac{\partial^n u(t,x)}{\partial t^n}+ +\sum_{\beta=0}^{n-1}\alpha_\beta A_\beta(D)\frac{\partial^\beta u(t,x)}{\partial t^\beta}=0, \tag{1}$$

$$u(t_q,x)=\varphi_q(x), \quad q=1,\dots,n,\ 0\le t_1<t_2<\dots<t_n\le T<\infty, \tag{2}$$

where

$$\alpha_\beta\in C,\ \beta=0,1,\dots,n-1;$$

$$\operatorname{Re}\alpha_0\neq 0;\ D=(D_1,\dots,D_p),\ D_j=\frac{1}{i}\frac{\partial}{\partial x_j},\quad j-1,\dots,p;$$

$A_\beta(D)$, $\beta=0,1,\dots,n$, are pseudodifferential operators, whose symbols $A_\beta(\xi)$, $\beta=0,1,\dots,n$, are real-valued functions, $k\in\mathbb{Z}^p$.

Below we use the notations:

$$x=(x_1,\dots,x_p)\in R^p,\ k=(k_1,\dots,k_p)\in Z^p,$$

$(k,x)=k_1x_1+\dots+k_px_p$, $|k|=|k_1|+\dots+|k_p|$, Γ' is the space of all continuous antilinear functionals over Γ', which coincides with the space of formal trigonometric series; $C^n([0,T],\Gamma)$ $(C^n([0,T],\Gamma'))$ is the space of functions $z(t,x)$ defined in the region $\overline{D}$, n times continuously differentiable in t, such that for each fixed number $t\in[0,T]$, $\partial^m z/\partial t^m\in\Gamma(\Gamma')$, $m=0,1,\dots,n$.

The action of the operator $A_\beta(D)$, $\beta=0,1,\dots,n$, on a periodic function of the form

$$\psi(x)=\sum_{|k|\ge 0}\psi_k\exp(i(k,x)) \tag{3}$$

is define as follows:

$$A_\beta(D)\psi(x)=\sum_{|k|\ge 0}A_\beta(k)\psi_k\exp(i(k,x)), \quad \beta=0,1,\dots,n. \tag{4}$$

Further, we assume that

$$(\forall k\in Z^p)\ A_n(k)\neq 0. \tag{5}$$

The solution of the problem (1), (2) we find in the form of a series

$$u(t,x)=\sum_{|k|\geq 0} u_k(t)\exp(i(k,x)). \quad (6)$$

Each of the coefficients $u_k(t), k\in Z^p$, is, respectively, the solution of the problem:

$$A_n(k)\frac{d^n u_k(t)}{dt^n}+\sum_{\beta=0}^{n-1}\alpha_\beta A_\beta(k)\frac{d^\beta u_\beta(t)}{dt^\beta}=0, \quad (7)$$

$$u_k(t_q)=\varphi_{qk},\ q=1,\dots,n, \quad (8)$$

where φ_{qk} are Fourier coefficients of the function $\varphi_q(x)$, $q=1,\dots,n.$

We denote by $\lambda_j(k)$, $j=1,\dots,n,$ the roots of the characteristic equation

$$A_n(\lambda,k)\equiv A_n(k)\lambda^n+\sum_{\beta=0}^{n-1}\alpha_\beta A_\beta(k)\lambda^\beta=0, \quad (9)$$

which corresponds to the equation (7). We assume that for each $k\in Z^p, \lambda_j(k), j=1,\dots,n,$ are distinct and different from zero.

Solution of the problem (7), (8) is represented by the formula

$$u_k(t)=\sum_{j=1}^{n} C_{kj}\exp(\lambda_j(k)t), \quad (10)$$

where C_{kj}, $j=1,\dots,n,$ is a solution of the corresponding system of linear algebraic equations

$$\sum_{j=1}^{n} C_{kj} exp(\lambda_j(k)t_q)=\varphi_{qk},\ q=1,\dots,n,$$

which determinant $\Delta(k)$ has the form

$$\Delta(k)\equiv\det\left\|\exp(\lambda_j(k)t_q)\right\|_{q,j=1}^{n}. \quad (11)$$

3. Uniqueness and existence of the solution.

Problem (1) - (2) can not have two different solutions if and only if the corresponding homogeneous problem has only the trivial solution.

Theorem 1 *For the uniqueness of the solution of the problem (1), (2) in the space* $(C^n([0,T],\Gamma'))$ *it is necessary and sufficient that:*

$$(\forall k\in Z^p)\quad \Delta(k)\neq 0. \quad (12)$$

The proof is carried out according to the theorem 1 proving scheme [1] and follows from the uniqueness of the expansion of functions from Γ' into Fourier series.

Further, we assume that condition (12) is satisfied. Then from (6) and (10) we obtain the formal presentation of the solution of the problem (1), (2) as a series

$$u(t,x)=\sum_{|k|\geq 0}\sum_{j,q=1}^{n}\frac{\Delta_{qj}(k)\varphi_{qk}\exp((\lambda_j k)t+i(k,x))}{\Delta(k)}, \quad (13)$$

where $\Delta_{qj}(k)$ – cofactor of the element $\exp(\lambda_j(k)t_q)$ of the determinant (11).

From the formula (13), taking into consideration that in the space Γ' an arbitrary trigonometric series is convergent we obtain the following statement.

Theorem 2. *Let the condition (12) is valid. If function* $\varphi_q\in\Gamma(\Gamma')$, $q=1,\dots,n,$ *then there exists a solution of the problem (1), (2), which belongs to the space* $C^n([0,T],\Gamma)$ $(C^n([0,T],\Gamma'))$.

Consider the particular case of the problem (1),(2) when

$$A_\beta(D)\equiv(1-\Delta)^m=\left(1+D_1^2+\ldots+D_p^2\right)^{m\beta}, \quad m_\beta\in R,\ \beta=0,1,\dots,n, \quad (14)$$

where $m_\beta>m_n$, $\beta=0,1,\dots,n-1,$ and conditions (2) fix the status of the process, which is described by equation (1) at regular intervals, i.e.

$$t_q=(q-1)t_0,\ q=1,\dots,n,\ t_0=T/(n-1). \quad (15)$$

Obviously, the operators' symbols satisfy the conditions

$$(\forall k\in Z^p)\ N_0\left(1+|k|^2\right)^{m\beta}\leq \leq|A_\beta(k)|\leq N_1\left(1+|k|^2\right)^{m\beta}, \quad (16)$$

$$\beta=0,1,\dots,n,\ 0<N_0\leq N_1.$$

The characteristic determinant of the problem (1),(2), (14), (15) is calculated by the formula

$$\Delta(k)=\prod_{1\leq p<r\leq n}\left(\exp(\lambda_r(k)t_0)-\exp(\lambda_p(k)t_0)\right). \quad (17)$$

Theorem 1 and formula (17) implies that for uniqueness of the solution of the considered problem it is necessary and sufficient that

$$(\lambda_p(k)-\lambda_p(k))t_0\neq 2\pi il, \quad l\in Z,\ k\in Z^p,\ 1\leq p<r\leq n. \quad (18)$$

By conditions (18) the solution of problem (1), (2),(14), (15) is formally represented as a series

$$u(t,x)= =\sum_{|k|\geq 0}\sum_{j,q=1}^{n}\frac{(-1)^{q-1}\varphi_{qk}S_{n-q}^{(j)}\exp(\lambda_j(k)t+i(k,x))}{\prod_{p=1,\,p\neq j}^{n}\exp(\lambda_j(k)t_0)-\exp(\lambda_p(k)t_0)}, \quad (19)$$

where $S_{(n-q)}^{(j)}$ is the sum of the products of elements $\exp\left(\lambda_j(k)t_0\right)$, $r=1,\dots,n$, $r\neq j$, $j=1,\dots,n$, taken for $(n-q)$ in each product, $S_0^{(j)}\equiv 1$, $j=1,\dots,n$.

Let us consider case $m<m_n$, where $m=\max\limits_{0\le\beta\le n-1}\{m_\beta\}$.

In this case, the roots $\lambda_j(k)$, $j=1,\dots,n$, satisfy the inequalities

$$\left|\lambda_j(k)\right|\le H_1\left(1+|k|^2\right)^{\gamma},\ j=1,\dots,n. \tag{21}$$

where $\gamma=-(m-m_n)/n$, $H_1>0$ is a positive constant, which does not depend on $k\in Z^p$.

Theorem 3 *Let condition (12) is valid and there exists positive constants such that for all (except a finite number of) vectors* $k\in Z^p$ *, the inequality*

$$\left|\Delta(k)\right|>|k|^{-\beta_1}, \tag{22}$$

is satisfied.

If $\varphi_q\in C^h(\Omega)$, $h=[n\gamma+\beta_1+m_n+p/2]+1$, $q=1,\dots,n$, *then there exists a solution of the problem (1), (2) from the space* $C^{(n,m_n)}(\bar{D})$, *that continuously depends on functions* $\varphi_q(x)$, $q=1,\dots,n$.

The proof. If functions $\varphi_q(x)$, $q=1,\dots,n$ satisfy the conditions of the theorem, then the inequalities

$$|\varphi_{qk}|\le \tilde{N}_1|k|^{-h}\|\varphi_q(x)\|_{C^h(\Omega)}, C_1>0. \tag{23}$$

are hold true.

From the estimates (21) (23) and formula (13) for norms of the solution of the problem (1), (2) we have the estimate

$$\|u\|_{C^{(n,m_n)}(\bar{D})}\le C_2\sum_{|k|>0}|k|^{-p-\varepsilon}\sum_{q=1}^{n}\left\|\varphi_q\right\|_{C^h(\Omega)}\le$$

$$\le C_3\sum_{q=1}^{n}\left\|\varphi_q\right\|_{C^h(\Omega)}. \tag{24}$$

From the inequality (24) follows the proving of the theorem.

4. Metrical estimations

Lemma 1. *Let condition (12) and evaluations (21) are fulfilled. Then for any fixed* $\mathrm{Im}\,\alpha_0,\alpha_1,\dots,\alpha_{n-1}$ *and almost all (with respect to Lebesgue measure in* R^p*) numbers* $\mathrm{Re}\,\alpha_0$ *inequality holds*

$$\prod_{1\le j<r\le n}\left|\lambda_r(k)-\lambda_j(k)\right|>M_1\left(1+|k|^2\right)^{-(m_n-m_0+p/2)/(n-1)/2}, \tag{25}$$

$$M_1>0,$$

for all (except for a finite number) vectors $k\in Z^p$.

Theorem 4. *For an arbitrary fixed* $\mathrm{Im}\,\alpha_0,\alpha_1,\dots,\alpha_{n-1}$ *and almost all (concerning the Lebesgue measure* R^{n+1}*) vectors* $(Re\alpha_0,\bar{t})$, *where* $\bar{t}=(t_1,\dots,t_n)$, *inequality (22) holds for all (except for a finite number of) vectors* $k\in Z^p$ *and* $\beta_1>(n-1)/2\left((n+1)p/2+m-m_0\right)$.

The proving is based on lemma 1 and carried out the scheme of prooving of the theorem 3 [1].

Let us consider case $m=m_n$.

Estimates (21) in this case have the form

$$\left|\lambda_q(k)\right|\le H_3, H_3>0, k\in Z^p, q=1,\dots,n. \tag{26}$$

Theorem 5. *Let* $m=m_n$, *and conditions (26) are hold true. If* $\varphi_q\in C^{h_1}(\Omega)$, $q=1,\dots,n$, $h_1=\left[p/4\left(n^2+1\right)+(m-m_0)(n-1)/2+m_n\right]+1$, *then for arbitrary fixed* $\mathrm{Im}\,\alpha_0,\ \alpha_1,\dots,\alpha_{n-1}$ *and almost all (concerning to Lebesgue measure in* R^{n+1}*) vectors* $(Re\alpha_0,\bar{t})$ *there exists the solution of the problem (1), (2) in the space* $C^{(n,m_n)}(\bar{D})$, *which continuously depends on the functions* $\varphi_q(x)$, $q=1,\dots,n$.

The proof. Note that for almost all (*concerning to Lebesgue measure* R^{n+1}) vectors $(Re\alpha_0,\bar{t})$, will get evaluation

$$\left|\Delta(k)\right|>|k|^{-(p/2(n+1)+m_n-m_0)(n-1)-\varepsilon}, \varepsilon>0. \tag{27}$$

From the estimates (26), (27) and formula (13) for the norm of the solution of the problem we obtaine the estimate

$$\|u\|_{C^{(n,m_n)}(\bar{D})}\le C_4\sum_{|k|>0}|k|^{-p-\varepsilon}\sum_{q=1}^{n}\left\|\varphi_q\right\|_{C^{h_1}(\Omega)}\le$$

$$\le C_5\sum_{q=1}^{n}\left\|\varphi_q\right\|_{C^{h_1}(\Omega)}. \tag{28}$$

From the inequality (28) follows theorem proving.

5. Conclusion

It was established conditions for the existence, uniqueness and continuous dependence from the right parts of the boundary conditions of the solution of the multipoint problem for linear equations with pseudodifferential operators.

The results complement scientific works that have been studied in [2-4]. They can be used in the study of specific practice problems which are modeled by means considered in the problem, and in further theoretical studies of problems with multipoint conditions for partial differential equations.

References

[1]. *Ptashnyk B.I.* Incorrect boundary value problems for differential equations. – K.: Scientific thought, 1984. – 264p.

[2.] *Klyus I.S., Ptashnyk B.I.* Multipoint problem with complex coefficients for partial differential equations not solved as to the highest derivative // Math. methods and physic.- mechanic. fields. – 1998. – 41, №4. – P. 83–88.

[3.] *Klyus I. S., Ptashnyk B.I.* Multipoint problem for partial differential equations with constant coefficients not solved as to the highest derivative // Bulletin of the state. Univ. "Lviv Polytechnic". Applied Mathematics.-1998. – 1, №337. – P. 112 – 115.

[4.] *Klyus I.S., Ptashnyk B.I.* Multipoint problem for partial differential equations not solved as to the highest derivative // Ukr. Math. Journ. – 1999. – 51, №12. – P. 1604–1613.

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І.С. Клюс[1], П.Б. Василишин[2], В.С. Коновалюк[3]. Багатоточкова задача для псевдодифференціальних рівнянь.

[1,3]Національний авіаційний університет, просп. Космонавта Комарова, 1, Київ, Україна, 03680

[2]Прикарпатський національний університет імені Василя Стефаника, вул. Шевченка 57, Івано-Франківськ, Україна,76025

E-mails:, [1]isklyus @gmail.com; [2]pbvasylyshyn@ukr.net; [3]Valentinakonovaliuk@gmail.com

Досліджено коректність задачі з багатоточковими умовами за часовою змінною та умовами періодичності за просторовими координатами для псевдодиференціальних рівнянь. Встановлено умови існування та єдиності розв’язку задачі. Доведено метричні теореми про оцінки знизу малих знаменників, які виникають при побудові розв’язку задачі.

Ключові слова: багатоточкові умови; диференціальні рівняння; малі знаменники; псевдодиференціальні оператори

И.С. Клюс[1], П.Б. Василишин[2], В.С. Коновалюк[3]. Многоточечная задача для псевдодифференциальных уравнений.

[1,3]Национальный авиационный университет, просп. Космонавта Комарова, 1, Киев, Украина, 03680

[2]Прикарпатский национальный университет имени Василия Стефаника, ул. Шевченко 57, Ивано-Франковск, Украина,76025

E-mails:, [1]isklyus @gmail.com; [2]pbvasylyshyn@ukr.net; [3]Valentinakonovaliuk@gmail.com

Исследована корректность задачи с многоточечными условиями по часовой переменной и периодическими условиями по пространственным координатам для псевддифференциальных уравнений. Установлены условия существования и единственности решения задачи. Доказаны метрические теоремы об оценках снизу малых знаменателей, которые возникают при построении решения задачи.

Ключевые слова: дифференциальные уравнения, малые знаменатели; многоточечные условия; псевдодиференциальные операторы

Klyus Iryna. PhD. Associate Professor.
National Aviation University, Kyiv, Ukraine.
Education: Drogobych Ivan Franko State Pedagogical Institute (1993).
Research area: differential equations.
Publications: 35.
E-mail: isklyus@gmail.com

Vasylyshyn Pavlo. PhD. Associate Professor.
Precarpathian National University named after Vasyl Stefanyk, Ivano-Frankivsk, Ukraine
Education: Precarpathian University named after Vasyl Stefanyk , Ivano-Frankivsk, Ukraine (1993).
Research area: Differential equations.
Publications: 29.
E-mail: pbvasylyshyn@ukr.net

Konovaliuk Valentyna. PhD. Associate Professor.
National Aviation University, Kyiv, Ukraine.
Education: Vinnytsa State Pedagogical Institute, Vinnytsa, Ukraine (1970).
Research area: Theory of probability and mathematical statistics, semi-Markovian processes.
Publications: 37.
E-mail: Valentinakonovaliuk@gmail.com